differential calculus problems with solution

Differential calculus problems with solution are essential for understanding how functions behave and change. Differential calculus is a branch of mathematics focused on the concept of the derivative, which represents the rate of change of a function. This article will explore various differential calculus problems, provide detailed solutions, and explain the underlying principles. Through examples, we will illustrate how to apply these concepts effectively.

Understanding Derivatives

Before diving into specific problems, it's crucial to understand what a derivative is. The derivative of a function $f(x)$ at a point x is defined as the limit:

$$
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
$$

This expression quantifies how the function $f(x)$ changes as x changes. In practical terms, the derivative can be interpreted as the slope of the tangent line to the curve represented by the function at a given point.

Basic Derivative Rules

Before solving problems, familiarize yourself with these fundamental derivative rules:

1. Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
2. Constant Rule: If $f(x) = c$ (where c is a constant), then $f'(x) = 0$.
3. Sum Rule: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
4. Difference Rule: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
5. Product Rule: If $f(x) = g(x) \cdot h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$.
6. Quotient Rule: If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$.
7. Chain Rule: If $f(g(x))$, then $f'(x) = f'(g(x)) \cdot g'(x)$.

Problem 1: Basic Derivative Calculation

Problem: Find the derivative of the function $f(x) = 3x^4 - 5x^2 + 7$.

Solution:

Using the power rule, we differentiate term by term:

1. For $3x^4$:
$$
f'(x) = 3 \cdot 4x^{4-1} = 12x^3
$$
2. For $-5x^2$:
$$
f'(x) = -5 \cdot 2x^{2-1} = -10x
$$
3. For the constant 7:
$$
f'(x) = 0
$$

Combining these results:
$$
f'(x) = 12x^3 - 10x
$$

Thus, the derivative of $f(x)$ is $f'(x) = 12x^3 - 10x$.

Problem 2: Applying the Product Rule

Problem: Differentiate the function $f(x) = (2x^2 + 3)(x^3 - 4)$.

Solution:

We will use the product rule, which states $f'(x) = g'(x)h(x) + g(x)h'(x)$ where $g(x) = 2x^2 + 3$ and $h(x) = x^3 - 4$.

1. Differentiate $g(x)$:
$$
g'(x) = 4x
$$
2. Differentiate $h(x)$:
$$
h'(x) = 3x^2
$$

Now we apply the product rule:
$$
f'(x) = g'(x)h(x) + g(x)h'(x)
$$
Substituting the values:
$$
f'(x) = (4x)(x^3 - 4) + (2x^2 + 3)(3x^2)
$$
Expanding each term:

1. $4x(x^3 - 4) = 4x^4 - 16x$
2. $(2x^2 + 3)(3x^2) = 6x^4 + 9x^2$

Combining these results:

$$
f'(x) = (4x^4 + 6x^4) + (-16x + 9x^2) = 10x^4 + 9x^2 - 16x
$$

Thus, the derivative is $f'(x) = 10x^4 + 9x^2 - 16x$.

Problem 3: Using the Quotient Rule

Problem: Differentiate the function $f(x) = \frac{x^2 + 1}{x - 2}$.

Solution:

We will use the quotient rule, which states $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$, where $g(x) = x^2 + 1$ and $h(x) = x - 2$.

1. Differentiate $g(x)$:

$$
g'(x) = 2x
$$

2. Differentiate $h(x)$:

$$
h'(x) = 1
$$

Applying the quotient rule:

$$
f'(x) = \frac{(2x)(x - 2) - (x^2 + 1)(1)}{(x - 2)^2}
$$

Now simplify the numerator:
1. $2x(x - 2) = 2x^2 - 4x$
2. Subtract $(x^2 + 1)$:

$$
2x^2 - 4x - x^2 - 1 = x^2 - 4x - 1
$$

Thus, the derivative simplifies to:

$$
f'(x) = \frac{x^2 - 4x - 1}{(x - 2)^2}
$$

Problem 4: Application in Real-Life Context

Problem: A company's profit P in thousands of dollars is given by the function $P(x) = -2x^2 + 40x - 100$, where x is the number of units sold in thousands. Find the rate of change of profit when 15 units are sold.

Solution:

First, find the derivative $P'(x)$:

$$P'(x) = -4x + 40$$

Next, evaluate the derivative at $x = 15$:

$$P'(15) = -4(15) + 40 = -60 + 40 = -20$$

This means that the rate of change of profit when 15 units are sold is -20 thousand dollars per unit. Therefore, the profit is decreasing at this rate.

Conclusion

In this article, we explored several differential calculus problems with solution that illustrate the fundamental concepts of derivatives, including the power rule, product rule, and quotient rule. These techniques are essential for analyzing functions and their rates of change in various contexts, from basic polynomial functions to real-world applications like profit maximization. Mastery of these principles not only enhances mathematical understanding but also equips individuals with the skills necessary to tackle complex problems in calculus and beyond.

Frequently Asked Questions

What is the derivative of the function f(x) = 3x^2 + 5x - 7?

The derivative f'(x) = 6x + 5.

How do you find the maximum or minimum value of a function using differential calculus?

To find the maximum or minimum, first find the derivative of the function, set it to zero to find critical points, then use the second derivative test to determine concavity.

What is the product rule in differential calculus?

The product rule states that if you have two functions u(x) and v(x), then the derivative of their product is given by (uv)' = u'v + uv'.

How can you solve the problem of finding the slope of the tangent line to the curve y = x^3 - 4x at x = 2?

First, find the derivative y' = 3x^2 - 4. Then evaluate it at x = 2: y'(2) = 3(2^2) - 4 = 12 - 4 = 8. The slope of the tangent line at that point is 8.

What is implicit differentiation and when is it used?

Implicit differentiation is used when you have an equation involving both x and y that is difficult to solve for y explicitly. You differentiate both sides with respect to x, treating y as a function of x.

Can you provide an example of applying the chain rule?

Sure! For the function f(x) = (2x^3 + 1)^4, use the chain rule: f'(x) = 4(2x^3 + 1)^3 (6x^2) = 24x^2(2x^3 + 1)^3.